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AN EXTENSION OF THE ERDŐS–STONE THEOREM

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Let $K_p(u_1,\dots,u_p)$ be the complete p-partite graph whose ith vertex class has u_i vertices $(1 \le i \le p)$. We show that the theorem of Erdős and Stone can be extended as follows. There is an absolute constant $\alpha > 0$ such that, for all $r \ge 1$, $0 < \gamma < 1$ and $0 < \varepsilon \le 1/r$, every graph $G = G^n$ of sufficiently large order $|G| = n$ with at least

$$\left(1 - \frac{1}{r} + \varepsilon\right)\binom{n}{2}$$

edges contains a $K_{r+1}(s, m, \dots, m, \ell)$, where $m = m(n) = \lfloor \alpha(1-\gamma)(\log n)/\log r \rfloor$, $s = s(n) = \lfloor \alpha(1-\gamma)(\log n)/r\log(1/\varepsilon)\rfloor$, and $\ell = \ell(n) = \lfloor \alpha\varepsilon^{1+\gamma/2}n^{\gamma}\rfloor$. The above result strengthens a sharpening of the Erdős–Stone theorem due to Bollobás, Erdős, and Simonovits, which guaranteed the existence of a $K_{r+1}(s,\dots,s)$ in G. The strengthening in our result lies in the fact that m above is independent of ε and ℓ can be demanded to be almost the first power of n. A related conjecture extending the Chvátal–Szemerédi sharpening of the Erdős–Stone theorem is presented.

1. Introduction

For a graph G let us denote its order by $|G|$ and its *size*, the number of edges of G, by $e(G)$; we shall usually write G^n for a graph of order n. Let $r \ge 1$ and $0 < \varepsilon \le 1/r$ be given. A celebrated theorem of Erdős and Stone [6] says that, for any fixed integer $t \ge 1$, if a graph $G = G^n$ has order $|G| = n \ge n_0(r, \varepsilon, t)$ and

$$e(G) \ge \left(1 - \frac{1}{r} + \varepsilon\right)\binom{n}{2}, \tag{1}$$

then G must contain a complete $(r+1)$-partite graph $K_{r+1}(t)$, whose every vertex class has cardinality t. This result is sharp in the following qualitative sense. Let $T_r(n)$ denote the complete r-partite graph of order n whose vertex classes have cardinality as equal as possible. Then obviously $T_r(n)$ contains no complete subgraph of order $r+1$ but

$$e(T_r(n)) = (1 + o(1))\left(1 - \frac{1}{r}\right)\binom{n}{2},$$

and hence ε in (1) has to be strictly positive. This very powerful and rather surprising result of Erdős and Stone has many important applications in extremal graph theory: see for instance [1], Chapter VI, Sections 3 and 4.

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Let us denote by $t(\varepsilon,r,n)$ the largest t for which every graph of order n and size $(1-1/r+\varepsilon)\binom{n}{2}$ must contain a $K_{r+1}(t)$. The Erdős–Stone theorem tells us that

$$t(\varepsilon,r,n)\to\infty \tag{2}$$

as $n\to\infty$; in fact their proof gives that $t(\varepsilon,r,n)\geq\sqrt{(\log^{(r)} n)}$, where $\log^{(r)}$ denotes the r times iterated logarithm. What is the real growth of $t(\varepsilon,r,n)$? Results concerning this question were obtained by Bollobás and Erdős [2], Bollobás, Erdős, and Simonovits [3], and Chvátal and Szemerédi [4]. It is proved in [3] that there is an absolute constant $c>0$ such that

$$t(\varepsilon,r,n)\geq\frac{c\log n}{r\log(1/\varepsilon)}, \tag{3}$$

and Chvátal and Szemerédi [4], with a rather involved proof based on the deep regularity lemma of Szemerédi [7], improved this to

$$t(\varepsilon,r,n)\geq\frac{\log n}{500\log(1/\varepsilon)}, \tag{4}$$

for large enough n. A result in [1] implies that (4) above is best possible up to the constant $1/500$.

Let a graph $G=G^n$ of order n be given, and assume that its edge-density $d(G)=e(G)\binom{n}{2}^{-1}$ is $d>0$. As d grows, what can we say about the various complete $(r+1)$-partite subgraphs $(r=1,2,\dots)$ that are guaranteed to exist in G? By the results given above, we first see that the critical values for d are $d_r=1-1/r$: when d increases beyond d_r, we are suddenly guaranteed to find complete $(r+1)$-partite subgraphs $K_{r+1}(t)$ in G. Also, the cardinality of the vertex classes of our $K_{r+1}(t)$ increases as $d-d_r$ increases, until we have $d=d_{r+1}$, when complete $(r+2)$-partite subgraphs are "born". Note that the above theorems only tell us that when d is just above d_r, we have a $K=K_{r+1}(t)$ in G but *all* its vertex classes are rather small if $d-d_r$ is small. Now, since $d-d_{r-1}>r^{-2}$ is large, there are complete r-partite graphs $K_r(t)$ in G with t large. A very natural question then is the following: is it true that if $d=d(G)$ is just above d_r and $|G|$ is sufficiently large, then G contains a complete r-partite subgraph such that (i) its vertex classes are of large cardinality, *i.e.* of cardinality *independent* of $\varepsilon=d-d_r$, and (ii) it can be extended to a complete $(r+1)$-partite subgraph whose smallest vertex class is as large as the right-hand side of (4), where $\varepsilon=d-d_r$? In other words, is it true that our G contains an $(r+1)$-partite graph all of whose vertex classes are of large cardinality (independent of $\varepsilon=d-d_r$) except for one, whose cardinality is as given in (4)?

We believe that the answer to the above question is "yes". In fact, it seems to us that the following extension of the Chvátal–Szemerédi theorem is very plausible. As usual, we write $K_p(u_1,\dots,u_p)$ for the complete p-partite graph whose vertex classes have cardinality $u_1,\dots,u_p$.

Conjecture 1. *There is an absolute constant $\alpha>0$ such that, for all $r\geq1$ and $0<\varepsilon\leq1/r$, every G^n of sufficiently large order satisfying*

$$e(G^n)\geq\left(1-\frac{1}{r}+\varepsilon\right)\binom{n}{2}$$

contains a $K_{r+1}(s_0,m_0,\dots,m_0)$, *where*

$$s_0 = s_0(n) = \left\lfloor \frac{\alpha \log n}{\log(1/\varepsilon)} \right\rfloor$$

and

$$m_0 = m_0(n) = \left\lfloor \frac{\alpha \log n}{\log r} \right\rfloor . \qquad \blacksquare$$

Our main aim in this note is to prove a somewhat different extension of the Erdős–Stone theorem and, for that matter, an extension of the Bollobás–Erdős–Simonovits theorem.

Theorem 2. *There is an absolute constant* $\alpha > 0$ *such that, for all* $r \geq 1$, $0 < \gamma < 1$, *and* $0 < \varepsilon \leq 1/r$, *every* G^n *of sufficiently large order satisfying*

$$e(G^n) \geq \left(1 - \frac{1}{r} + \varepsilon\right)\binom{n}{2}$$

contains a $K_{r+1}(s_1,m_1,\dots,m_1,\ell_1)$, *where*

$$s_1 = s_1(n) = \left\lfloor \alpha(1-\gamma)\frac{\log n}{r\log(1/\varepsilon)} \right\rfloor ,$$

$$m_1 = m_1(n) = \left\lfloor \alpha(1-\gamma)\frac{\log n}{\log r} \right\rfloor ,$$

and

$$\ell_1 = \ell_1(n) = \lfloor \alpha \varepsilon^{1+\gamma/2} n^{\gamma} \rfloor .$$

Note that s_1 equals s_0/r up to an absolute multiplicative constant, and hence Theorem 2 comes quite close to proving Conjecture 1. Furthermore, this theorem shows that, perhaps a little unexpectedly, the largest vertex class of our $(r+1)$-partite graph can be guaranteed to be extremely large, namely of order at least $n^{1-\rho}$ for any fixed $\rho > 0$. Note that Conjecture 1 and Theorem 2 are incomparable in the sense that neither implies the other. However, it is possible that Conjecture 1 and Theorem 2 have a common extension, namely that Theorem 2 also holds if we change s_1 to rs_1 and ℓ_1 to $\lfloor c(r,\gamma,\varepsilon)n^{\gamma} \rfloor$, where $c(r,\gamma,\varepsilon) > 0$.

Let us remark that our proof of Theorem 2 is based on a simple inequality, given in Lemma 5 below. Indeed, with the help of this lemma the proof of (3) by Bollobás, Erdős, and Simonovits can be easily upgraded to a proof of Theorem 2. We expect that to prove Conjecture 1 and its extension stated above a completely new approach is needed.

This note is organised as follows. In the next section we prove Theorem 2 in the particular case where $r = 1$ and, by applying a standard lemma, we note that for $r \geq 2$ we can derive Theorem 2 as a corollary of a seemingly weaker result, Theorem 4. In the final section we formulate our key inequality and give its proof; we then close the note by proving Theorem 4.

2. Preliminaries

We shall now start the proof of Theorem 2. For simplicity, we shall not specify the value of α now: it will be clear that the inequalities we shall need in the sequel are valid if α is not larger than a certain absolute constant. Let us deal with the case $r=1$ separately first. This case will present no difficulty; we shall merely need very well-known methods (cf. [1], Chapter VI).

Let a graph $G=G^n$ of order n be given. Assume that $e(G)\geq\varepsilon\binom{n}{2}$ and that n is sufficiently large, so that our inequalities below are valid. Moreover, we may assume that $\varepsilon\leq 1/2$. Let $s=s(n)=\lfloor(1-\gamma)(\log n)/2\log(1/\varepsilon)\rfloor$. We shall show that G contains a copy of $K_2(s,\lceil n^\gamma\rceil)$.

For a vertex $x\in G$ let us denote its degree by $d(x)=d_G(x)$. The number of stars $K_2(1,s)$ present in G is

$$\sum_{x\in G}\binom{d(x)}{s}\geq n\binom{2e(G)/n}{s},$$

where the inequality follows from the convexity of the function $f_s(x)=\binom{x}{s}$. Hence, there exists a collection of s vertices of G that is joined to at least

$$\sum_{x\in G}\binom{d(x)}{s}\binom{n}{s}^{-1}\geq n\binom{\varepsilon(n-1)}{s}\binom{n}{s}^{-1}\geq\left(\frac{\varepsilon(n-1)-s+1}{n-s+1}\right)^s\geq n^\gamma$$

vertices, and this gives us a $K_2(s,\lceil n^\gamma\rceil)$ in G, as required.

Now we proceed to the case $r\geq 2$. We shall need the following standard lemma, whose proof may be found in [1], Chapter VI, p. 330. As usual, we write $\delta(G)$ for the minimal degree of a graph G.

Lemma 3. *Let $0<\varepsilon\leq d<1$. Suppose $n>4d/\varepsilon$ and $d(G^n)=e(G)\binom{n}{2}^{-1}>d$. Then G^n contains a subgraph H of order $n'>(\varepsilon/2)^{1/2}n$ such that $\delta(H)>(d-\varepsilon)n'$.* ∎

The above lemma tells us that the following theorem implies (and so is practically equivalent to) Theorem 2.

Theorem 4. *There is an absolute constant $\beta>0$ such that, for all $r\geq 2$, $0<\gamma<1$, and $0<\varepsilon\leq 1/r$, every G^n of sufficiently large order satisfying*

$$\delta(G^n)\geq\left(1-\frac{1}{r}+\varepsilon\right)n$$

contains a $K_{r+1}(s,m,\ldots,m,\ell)$, where

$$s=s(n)=\left\lfloor\beta(1-\gamma)\frac{\log n}{r\log(1/\varepsilon)}\right\rfloor,$$

$$m=m(n)=\left\lfloor\beta(1-\gamma)\frac{\log n}{\log r}\right\rfloor,$$

and

$$\ell=\ell(n)=\lfloor\beta\varepsilon n^\gamma\rfloor.$$

3. The proof of Theorem 4

We shall assume throughout this section that n is large enough. Let us once and for all fix $r\geq 2$, $0<\gamma<1$, and $0<\varepsilon\leq 1/r$. We assume that s, m and ℓ are as in Theorem 4, where $\beta>0$ will be specified later: it will be clear that all inequalities used in the sequel are valid provided $n\geq n_0=n_0(r,\gamma,\varepsilon)$, and $\beta\leq\beta_0$, where β_0 is a positive absolute constant. Let us start by assuming that $\beta>0$ is small enough so that if $t_0=\lfloor(\log n)/500\log r\rfloor$ then $s\leq t_0/10r$ and $m\leq t_0/12$.

Before we proceed, let us give an inequality that will be a crucial ingredient in deducing Theorem 4. The proof of this inequality, although standard, is rather cumbersome. Given two integer vectors $\mathbf{p}=(p_i)_1^r$ and $\mathbf{d}=(d_i)_1^r\in\mathbb{N}^r$ with $p_r\geq s$ and $p_i\geq m$ $(1\leq i<r)$, let us set

$$f(\mathbf{d})=f_{\mathbf{p}}(\mathbf{d})=\binom{d_r}{s}\prod_{i=1}^{r-1}\binom{d_i}{m}\Bigg/\binom{p_r}{s}\prod_{i=1}^{r-1}\binom{p_i}{m}.$$

Lemma 5. *Let $\mathbf{p}=(p_1,\dots,p_r)\in\mathbb{N}^r$ be an integer vector such that $p=(1/r)\sum_i p_i$ satisfies $t_0=\lfloor(\log n)/500\log r\rfloor\leq p\leq 3s/\varepsilon$, and $p_i\leq p+s$ for all $1\leq i\leq r$. Then, if*

$$S=\{\mathbf{d}=(d_i)_1^r\in\mathbb{N}^r: s\leq d_i\leq p_i \text{ all } i, \textstyle\sum_i d_i\geq(r-1)p, \text{ and } d_r=\min_i d_i\},$$

we have that $\min\{f(\mathbf{d})=f_{\mathbf{p}}(\mathbf{d}):\mathbf{d}\in S\}\geq n^{\gamma-1}$.

Proof. Note that if $\mathbf{d}=(d_i)_1^r\in S$ then $\min\{d_i:1\leq i\leq r-1\}\geq p/3$. Indeed, if we assume that $d_{r-1}\leq d_j$ for all $1\leq j\leq r-2$ then, since $\sum_i d_i\geq(r-1)p$, $d_i\leq p_i\leq p+s$, and $p\geq t_0$, we have that $d_{r-1}+d_r\geq(r-1)p-(r-2)(p+s)\geq 2p/3$. Hence $d_{r-1}\geq p/3$ since $d_{r-1}\geq d_r$. With a similarly simple argument we may check that $p_i\geq m$ $(1\leq i\leq r)$. For $\mathbf{d}=(d_i)_1^r$, set

$$g(\mathbf{d})=\prod_{i=1}^{r-1}\binom{d_i}{m}.$$

Let $\mathbf{d}=(d_i)_1^r$ be given, and assume that $d_k\geq d_\ell$ for some $1\leq k,\ell<r$. Define $\mathbf{d}'=(d_i')_1^r$ by putting

$$d_i'=\begin{cases} d_k+1 & \text{if } i=k\\ d_\ell-1 & \text{if } i=\ell\\ d_i & \text{otherwise,}\end{cases}$$

and then note that $g(\mathbf{d}')\leq g(\mathbf{d})$. Indeed,

$$\frac{g(\mathbf{d}')}{g(\mathbf{d})}=\frac{d_k+1}{d_k-m+1}\cdot\frac{d_\ell-m}{d_\ell}=\left(1-\frac{m}{d_\ell}\right)\Bigg/\left(1-\frac{m}{d_k+1}\right)\leq 1.$$

Hence $\min\{f(\mathbf{d}):\mathbf{d}\in S\}\geq f(\mathbf{d}^*)$, where $\mathbf{d}^*=(d_i^*)_1^r$ is such that $d_i^*=p_i$ for $1\leq i\leq r-3$, d_{r-1}^*, $d_{r-2}^*\geq p/3$, and $d_r^*=d_r$. Therefore we have that for all $\mathbf{d}=(d_i)_1^r\in S$

$$\begin{aligned} f(\mathbf{d}) &\geq \binom{d_r}{s}\binom{p/3}{m}^2\Bigg/\binom{p_r}{s}\binom{p_{r-2}}{m}\binom{p_{r-1}}{m}\\ &\geq \binom{p/3}{m}^2\Bigg/\binom{p+s}{s}\binom{p+s}{m}^2\geq\left(\frac{p/3-m}{p+s}\right)^{2m}\Bigg/\left(\frac{2pe}{s}\right)^s, \end{aligned}\tag{5}$$

since clearly $p+s \le 2p$. Now, as $p/3-m \ge p/4$,

$$\left(\frac{p/3-m}{p+s}\right)^{2m} \ge \left(\frac{1}{8}\right)^{2m} \ge 2^{-6\beta(1-\gamma)(\log n)/\log r} \ge n^{-(1-\gamma)/2}.$$

Also,

$$\left(\frac{2pe}{s}\right)^{s} \le \left(\frac{6e}{\varepsilon}\right)^{s} \le \left(\frac{6e}{\varepsilon}\right)^{\beta(1-\gamma)(\log n)/r\log(1/\varepsilon)}$$
$$\le \left[n(6e)^{(\log n)/\log r}\right]^{\beta(1-\gamma)/r} \le n^{(1-\gamma)/2}.$$

Hence we have that the right-hand side of (5) is bounded from below by $n^{-(1-\gamma)}$, as required. ∎

Let us now start the proof of Theorem 4. Let $G=G^n$ be so that

$$\delta(G) \ge \left(1-\frac{1}{r}+\varepsilon\right)n.$$

We shall need the following simple lemma in the sequel; its proof may be found in [1], Chapter VI, p. 332.

Lemma 6. *Let $X \subset G=G^n$ be a subset of the vertices of G, and let $Y \subset G-X$ be the set of the vertices of $G-X$ that are adjacent to at least $(1-1/r+\varepsilon/2)|X|$ vertices in X. Then*

$$|Y| > \frac{r\varepsilon n}{2} - |X|. \quad ∎$$

By the theorem of Chvátal and Szemerédi [4], we know that our graph $G=G^n$ contains a $K_r(t_0)=K_r(t_0,\dots,t_0)$, where $t_0=\lfloor(\log n)/500\log r\rfloor$. Amongst all complete r-partite subgraphs of G, let us choose $K=K_r(p_1,\dots,p_r)\subset G$ such that if $p=(1/r)\sum_i p_i$ then $p_i \le p+s$ for all i, and such that p is maximal under these conditions. Note that $p \ge t_0$. Let $\mathbf{p}=(p_i)_1^r$ and $P=\lceil 2s/\varepsilon\rceil$. The rest of the proof is now divided into two cases, according to the size of p.

Case 1. $p \ge P$

Note that we may assume that $P \le p \le P+1$. Let Z be the vertices of $G-V(K)$ that are joined to at least

$$\left(1-\frac{1}{r}+\frac{\varepsilon}{2}\right)rp = p(r-1)+\frac{\varepsilon pr}{2} > p(r-1)$$

vertices of K. Then by Lemma 6 we have that $|Z| > \varepsilon rn/2 - o(n) \ge \varepsilon rn/3$. Let z be a vertex in Z and let $\mathbf{d}_z$ be the sequence $(d_i^z)_1^r=(d_i(z))_1^r$, where $d_i(z)$ is the number of vertices in the ith class of K that are adjacent to z. For $z \in Z$, let N_z be the number of subgraphs $K' \subset K$ isomorphic to $K_r(s,m,\dots,m)$ such that (i) the first vertex class of K' is contained in the rth vertex class of K, and (ii) z is adjacent to all vertices of K'. Then

$$N_z \ge \binom{d_r^z}{s}\prod_{i=1}^{r-1}\binom{d_i^z}{m}.$$

Also, the number N of r-partite subgraphs $K'' \subset K$ isomorphic to $K_r(s,m,\ldots,m)$ with their first vertex classes contained in the rth vertex class of K equals

$$\binom{p_r}{s}\prod_{i=1}^{r-1}\binom{p_i}{m}.$$

Now let $Z_j = \{z \in Z : d_j(z) = \min_i d_i(z)\}$ $(1 \le j \le r)$. We may assume that $|Z_r| \ge |Z|/r$. Note that if $z \in Z_r$, then

$$d_r^z = \min_i d_i^z \ge p(r-1) + \frac{\varepsilon r p}{2} - (r-1)(p+s) \ge \frac{\varepsilon r p}{2} - (r-1)s.$$

As $p \ge P$,

$$\frac{1}{2}\varepsilon p \ge \frac{1}{2}\varepsilon P = \frac{1}{2}\varepsilon\left\lceil\frac{2s}{\varepsilon}\right\rceil \ge s.$$

Hence $d_r^z \ge \varepsilon rp/2 - (r-1)s \ge s$. Thus if $z \in Z_r$ then $\mathbf{d}_z \in S$, where S is as in Lemma 5. Therefore there is a copy of $K_r(s,m,\ldots,m)$ in K that is joined to at least $|Z_r|\min\{f_{\mathbf{p}}(\mathbf{d}) : \mathbf{d} \in S\}$ vertices in Z_r, and by Lemma 5 this quantity is at least $(\varepsilon n/3)/n^{1-\gamma} \ge \beta\varepsilon n^{\gamma}$. This completes the proof of this case.

Case 2. $p < P$

Let $Z \subset G - V(K)$ be as in the previous case, and let $U \subset Z$ be the set of the vertices in Z adjacent to at least s vertices in each class of K. Here we split our argument into two subcases, according to the size of U.

(i) Assume that $|U| \ge \varepsilon rn/12$. Then, again using Lemma 5, we see as in Case 1 that there is a $K_r(s,m,\ldots,m) \subset K$ joined to at least $\beta\varepsilon n^{\gamma}$ vertices of $G - V(K)$, completing the proof.

(ii) Assume now that $|U| < \varepsilon rn/12$. Our aim here is to show that this cannot happen by deriving a contradiction. Let $W = Z \backslash U$. By Lemma 6, we have that $|W| \ge |Z| - \varepsilon rn/12 \ge \varepsilon rn/4$. Let us classify the vertices in W by their neighbourhood in K: let us say that two vertices x and y $(x,\, y \in W)$ are *K-equivalent* if the set of vertices of K that are adjacent to x equals the corresponding set of vertices for y. How many classes of vertices have we split W into? Our aim now is to show that the number of such classes is small, and therefore there is a large class.

Let the C_i be the ith class of K. For each $w \in W$ let $i(w)$ be so that w is adjacent to fewer than s vertices in $C_{i(w)}$. Note that the number of vertices μ in $K - C_{i(w)}$ that are not adjacent to a fixed vertex w of W is less than $(r-1)(p+s) - ((r-1)p - s) = rs$. Hence the number of classes that we have split W into is at most

$$\sum_{i=1}^{r}\left\{\sum_{\lambda<s}\binom{p_i}{\lambda}\sum_{\mu<rs}\binom{rp-p_i}{\mu}\right\} \le 4r\binom{p+s}{s}\binom{rp}{rs} \le 4r2^s\left(\frac{p\mathrm{e}}{s}\right)^{s(1+r)}$$

$$\le 4r2^s\left(\frac{2\mathrm{e}+o(1)}{\varepsilon}\right)^{s(1+r)} \le 4r\left[(4\mathrm{e})^{(\log n)/\log(1/\varepsilon)}n\right]^{\beta(1-\gamma)(1+1/r)} \le 4rn^{1/2}.$$

Since W is large enough, there is a set $W' \subset W$ with cardinality $\lfloor p+s \rfloor$ consisting of K-equivalent vertices.

We shall now finish the proof by constructing $K' = K_r(p_1,\dots,p_r) \subset G$ for which $p_i \le p'+s$ $(1 \le i \le r)$, where $p' = (1/r)\sum_i p_i > p$. Note that this contradicts the maximality of p, and hence we are done. Denote by N_i the set of vertices of C_i that are joined to the vertices of W'. We may assume that $|N_1| \le \dots \le |N_r|$, and so in particular $|N_1| < s$.

If $|N_2| \le p$ let the classes of $K' = K_r(p_1,\dots,p_r)$ be given by

$$C_1' = W', \quad C_2' = N_1 \cup N_2, \quad \text{and} \quad C_j' = N_j \ (3 \le j \le r).$$

Since $\left|\bigcup_1^r N_j\right| > (r-1)p$, we have that $p' > p$. Note also that $|C_i'| \le p+s < p'+s$ for all i, and hence our K' contradicts the maximality of p.

If $|N_2| > p$ select $q = \lfloor p+1 \rfloor$ vertices from W' and also from each N_j $(2 \le j \le r)$. These r sets of vertices determine a copy of $K_r(q)$ in G, which contradicts the maximality of p. This completes the proof of Case 2, and hence of Theorem 4.

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